Advt. No. 04/2023 D. of Exam – 30/09/2023

23·03·2023

INSPS/TDD/IV/23

00218

MATHEMATICS

Paper—III

Full Marks : 100

Time : 3 hours

The figures in the margin indicate full marks for the questions

The symbols have their usual meanings

Answer *any* **ten** questions

1. If by a transformation from one set of rectangular axes to another with the same origin the expression $ax^2 + 2hxy + by^2$ changes to $a'x'^2 + 2h'x'y' + b'y'^2$, then prove that $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$. 10

2. What do you mean by a cyclic group? Prove that every cyclic group is Abelian. Is the group S_3 cyclic? Justify your answer. 2+5+3=10

3. *(a)* If p is a prime, then prove that $(p-1)! \equiv -1 (\text{mod } p)$. 6

(b) Using the properties of congruences, show that 41 divides $2^{20} - 1$. 4

4. *(a)* If z_1 and z_2 are two complex numbers, then prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ if and only if $\frac{z_1}{z_2}$ is purely imaginary. 5

(b) If $p = \cos a + i \sin a$ and $q = \cos b + i \sin b$, then show that

$$\frac{p-q}{p+q} = i \tan\left(\frac{a-b}{2}\right)$$ 5

5. A particle moves from rest in a straight line under an attractive force $\frac{\mu}{(\text{distance})^2}$. Show that if initial distance is $2a$, the distance will be a after the time

$$\left[\frac{\pi}{2}+1\right]\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}$$ 10

6. Define (a) integral domain and (b) prime ideal of a commutative ring. Let R be a commutative ring with unity and A be an ideal of R. Prove that $\frac{R}{A}$ is an integral domain if and only if A is prime. 3+7=10

7. Evaluate : 5+5=10

(a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$

(b) $\oint_C \frac{zf'(z)}{f(z)} dz$, where C is the circle $|z| = 5$ and $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$

8. Let F be the field of complex numbers and let T be a function from F^3 into F^3 defined by $T(x, y, z) = (x - y + 2z, 2x + y - z, -x - 2y)$. Verify that T is linear. Describe the null space of T. Is T invertible? If possible, find $T^{-1}(x, y, z)$. 4+3+1+2=10

9. Show that the functions $e^x \cos x$ and $e^x \sin x$ are linearly independent. Form a differential equation of second order having these two functions as independent solutions. 5+5=10

10. Reduce the differential equation $y'' + Py' + Qy = R$ to normal form, hence solve $(y'' + y)\cot x + 2(y' + y \tan x) = \sec x$. 5+5=10

11. Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic and find v such that $f(z) = u + iv$ is analytic. 5+5=10

12. Consider the function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}; & \text{if } x^2 + y^2 \neq 0 \\ 0 \quad ; & \text{if } x = 0 = y \end{cases}$$

Show that the function is discontinuous at origin, but f_x and f_y exist and are derivatives everywhere including origin. 5+5=10

13. The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1500 years.

(a) What percentage of the original radioactive nuclei will remain after 4500 years?

(b) In how many years will only one-tenth of the original number remain? (Take $\log 2 = 0{\cdot}301$) 7+3=10

14. A bakery produces two types of cookies—chocolate chip and caramel. The bakery anticipates daily demand for a maximum of 80 caramel cookies and 120 chocolate chip cookies. Due to a lack of raw materials and labour, the bakery can produce 120 caramel cookies and 140 chocolate chip cookies daily. For the bakery to be viable, it must sell a minimum of 240 cookies each day. Every chocolate chip cookie served generates ₹ 75 in profit, whereas each caramel cookie generates ₹ 88. Using graphical method, find the solution to the number of chocolate chip and caramel cookies that the bakery must produce each day to maximize profit. 5+5=10

★★★